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DIOPHANTINE ANALYSIS.

82. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

In the series $1^3 + 3^3 + 5^3 \dots$ find n so that the n th term and the sum of n terms shall both be squares.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The conditions of the problem require $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2-1) = \square$, and $(2n-1)^3 = \square$.

Whence it follows that $2n-1 = \square$, and $2n^2-1 = \square$.

$2n-1 = \square$, when $n = r^2 + (r-1)^2$, as 1, 5, 13, 25, 41, etc.

Hence $2n^2-1 = 2[r^2 + (r-1)^2]^2 - 1 = \square$.

Whence $2[r^2 + (r-1)^2]^2 = \square + 1$, two times a square equals the sum of two squares, the general formula for which is $2(p^2 + q^2)^2 = [(p+q)^2 - 2q^2]^2 + [(p-q)^2 - 2q^2]^2$.

Then $p^2 + q^2 = r^2 + (r-1)^2 \dots (1)$, and $(p-q)^2 - 2q^2 = \pm 1 \dots (2)$.

From (1), put $p=r$ and $q=(r-1)$; and substituting in (2) we obtain $1 - 2(r-1)^2 = \pm 1$, or $2(r-1)^2 = 0$ or 2 .

Whence $r-1 = 0$ or ± 1 , and $r = 1$ or 2 or 0 .

Substituting these values of r in $n = r^2 + (r-1)^2$, we find $n = 1$ and 5 , apparently the only integral values.

When $n = 1$, $2n-1 = 1$, $n^2(2n^2-1) = 1$, and $(2n-1)^3 = 1$.

When $n = 5$, $2n-1 = 9 = 3^2$, $n^2(2n^2-1) = 1^3 + 3^3 + 5^3 + 7^3 + 9^3 = 35^2$, and $(2n-1)^3 = (3^2)^3 = (3^3)^2 = 27^2$.

Also solved by G. B. M. ZERR, J. SCHEFFER, COOPER D. SCHMITT, and the PROPOSER.

AVERAGE AND PROBABILITY.

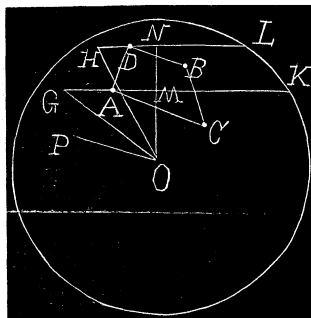
88. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

Solution by the PROPOSER.

Let GK be the diameter of the section of the sphere made by a plane through the three random points A, B, C ; HL the diameter of a parallel section through the fourth random point D ; M, N the centers of these sections, respectively; O the center of the sphere; OP a line such that AB is always parallel to the plane MOP .

Let $OG = OH = r$, $MA = x$, $AB = y$, $AC = z$, $DN = u$, $\angle GOM = \theta$, $\angle BAM = \varphi$, $\angle CAM = \psi$, $\angle HON = \beta$, $\angle MOP = \lambda$, and the angle the plane POM makes with a fixed plane through $OP = \rho$.



An element of the sphere at A is $r \sin \theta d\theta 2\pi x dx$; at B , $y^2 dy d\varphi d\lambda$; at C , $\sin(\varphi + \psi) \sin \lambda z^2 dz d\psi d\rho$; at D , $r \sin \beta d\beta 2\pi u du$.

The limits of θ are 0 and π ; of β , 0 and θ ; of x , 0 and $r \sin \theta = x'$ and triplicated; of u , 0 and $r \sin \beta = u'$ and sextupled; of φ , $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ , $-\varphi$ and $\frac{1}{2}\pi$; of λ , 0 and π ; of ρ , 0 and π ; of y , 0 and $2x \cos \varphi = y'$; of z , 0 and $2x \cos \psi = z'$.

The area of the triangle $ABC = \frac{1}{2} y z \sin(\varphi + \psi)$.

Altitude of tetrahedron $= r(\cos \beta - \cos \theta)$.

Volume of tetrahedron $= \frac{1}{6} r y z \sin(\varphi + \psi)(\cos \beta - \cos \theta)$.

Since the whole number of ways four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the required average triangle is

$$\begin{aligned}
 \Delta &= \frac{729}{128\pi^4 r^{12}} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^\pi \int_0^\pi \int_0^{y'} \int_0^{z'} \frac{1}{6} r y z \sin(\varphi + \psi) \\
 &\quad \times (\cos \beta - \cos \theta) r \sin \theta d\theta 2\pi x dx r \sin \beta d\beta 2\pi u du \sin(\varphi + \psi) d\varphi d\psi \sin \lambda d\lambda d\rho y^2 dy z^2 dz \\
 &= \frac{243}{16\pi^2 r^9} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^\pi \int_0^\pi \int_0^{y'} \sin^3 \theta x^5 \sin \beta u \sin^2(\varphi + \psi) \cos^4 \psi \\
 &\quad \times \sin \lambda (\cos \beta - \cos \theta) d\theta d\beta dx du d\varphi d\psi d\lambda d\rho y^3 dy \\
 &= \frac{243}{4\pi^2 r^9} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^\pi \int_0^\pi \sin^3 \theta \sin \beta (\cos \beta - \cos \theta) x^9 u \sin^2 \\
 &\quad \times (\varphi + \psi) \cos^4 \varphi \cos^4 \psi \sin \lambda d\theta d\beta dx du d\varphi d\psi d\lambda d\rho \\
 &= \frac{243}{2\pi r^9} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^3 \theta \sin \beta (\cos \beta - \cos \theta) x^9 u \sin^2(\varphi + \psi) \\
 &\quad \times \cos^4 \varphi \cos^4 \psi d\theta d\beta dx du d\varphi d\psi \\
 &= \frac{81}{64\pi r^9} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^3 \theta \sin \beta (\cos \beta - \cos \theta) x^9 u (15\pi \cos^4 \varphi \\
 &\quad - 12\pi \cos^6 \varphi + 30\pi \cos^4 \varphi - 24\pi \cos^6 \varphi - 4\sin \varphi \cos^7 \varphi + 30\sin \varphi \cos^5 \varphi) d\theta d\beta dx du d\varphi \\
 &= \frac{1215\pi}{256 r^9} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^{u'} \sin^3 \theta \sin \beta (\cos \beta - \cos \theta) x^9 u d\theta d\beta dx du \\
 &= \frac{1215\pi}{512 r^7} \int_0^\pi \int_0^\theta \int_0^{x'} \sin^3 \theta \sin^3 \beta (\cos \beta - \cos \theta) x^9 d\theta d\beta dx \\
 &= \frac{243\pi r^3}{1024} \int_0^\pi \int_0^\theta \sin^{11} \theta \sin^3 \beta (\cos \beta - \cos \theta) d\theta d\beta \\
 &= \frac{81\pi r^3}{4096} \int_0^\pi \sin^{11} \theta (8 - 4\sin^2 \theta - \sin^4 \theta - 8\cos \theta) d\theta = \frac{36\pi r^3}{715}.
 \end{aligned}$$